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Unified Field Theory From Enlarged Transformation Group. The Consistent Hamiltonian

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A theory has been presented previously in which the geometrical structure of a real four-dimensional space time manifold is expressed by a real orthonormal tetrad, and the group of diffeomorphisms is replaced by a larger group. The group enlargement was accomplished by including those transformations to anholonomic coordinates under which conservation laws are covariant statements. Field equations have been obtained from a variational principle which is invariant under the larger group. These field equations imply the validity of the Einstein equations of general relativity with a stress-energy tensor that is just what one expects for the electroweak field and associated currents. In this paper, as a first step toward quantization, a consistent Hamiltonian for the theory is obtained. Some concluding remarks are given concerning the need for further development of the theory. These remarks include discussion of a possible method for extending the theory to include the strong interaction.

KEY WORDS: gravitation; electroweak; strong; unification.

1. INTRODUCTION

In Sections 1 and 2, we describe a theory in which the classical (unquantized) gravitational and electroweak fields appear as manifestations of geometrical structure in a real four-dimensional space–time manifold. In Section 3, we obtain the Hamiltonian for the theory as a first step toward quantizing the theory. In Section 4, we make some concluding remarks concerning the further development of the theory. One of these remarks suggests a method for extending the theory to include the strong interaction. [*Note*: In several prior papers, one of us (Pandres, 1981, 1984a,b, 1995, 1998, 1999), has based the theory, not on a manifold, but on a space in which paths, rather than points are the primary elements. In this paper, however, we show that the theory can be based entirely on a manifold.]

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It is well known that any general relativistic metric $g_{\mu\nu}$ may be expressed in terms of an orthonormal tetrad of vectors h^i_μ. The expression is

$$g_{\mu\nu} = g_{ij} h^i_\mu h^j_\nu \tag{1}$$

where $g_{ij} = g^{ij} = \text{diag}(-1, 1, 1, 1)$, and the summation convention has been adopted. Indices take the values 0,1,2,3, and $g^{\mu\nu}$ is defined by $g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha$, where δ^μ_α is the Kronecker delta. Latin (tetrad) indices are raised and lowered by using g^{ij} and g_{ij}, just as Greek (space time) indices are raised and lowered by using $g^{\mu\nu}$ and $g_{\mu\nu}$. Partial differentiation is denoted by a comma. Covariant differentiation with respect to the Christoffel symbol $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$ is denoted by a semicolon.

1.1. Motivation

We recall (Pandres, 1962, 1999) an argument which is a generalization of the "elevator" argument that led Einstein from special relativity to general relativity. The special relativistic equation of motion for a free particle is

$$\frac{d^2 x^i}{ds^2} = 0, \tag{2}$$

where $-ds^2 = g_{ij} dx^i dx^j$. Consider the image-equation of this free-particle equation under the transformation

$$dx^i = h^i_\mu dx^\mu \tag{3}$$

where the curl $f^i{}_{\mu\nu} = h^i{}_{\nu,\mu} - h^i{}_{\mu,\nu}$ is not zero. Equation (3) establishes a one-to-one correspondence between *coordinate increments* dx^i and dx^μ. Since $h^i{}_{\nu,\mu} - h^i{}_{\mu,\nu}$ is not zero, we cannot integrate Eq. (3) to get a one-to-one correspondence between coordinates x^i and x^μ. However, it follows from Eq. (3) that $\frac{dx^i}{ds} = h^i_\mu \frac{dx^\mu}{ds}$. Upon differentiating this with respect to s, using the chain rule, and multiplying by h^α_i, we see that Eq. (2) may be written

$$\frac{d^2 x^\alpha}{ds^2} + h^\alpha_i h^i{}_{\mu,\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \tag{4}$$

We follow Eisenhart (1925) in defining Ricci rotation coefficients by $\gamma^i{}_{\mu,\nu} = h^i{}_{\mu;\nu} = h^i{}_{\mu,\nu} - h^i_\sigma \Gamma^\sigma{}_{\mu\nu}$. Multiplication by h^α_i gives $h^\alpha_i h^i{}_{\mu,\nu} = \Gamma^\alpha{}_{\mu\nu} + \gamma^\alpha{}_{\mu\nu}$, and upon using this in Eq. (4) we have

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -\gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}. \tag{5}$$

The relation $\gamma_{\mu\nu i} = h^j_\mu \gamma_{j\nu\alpha} h^\alpha_i$ illustrates our general method for converting between Greek and Latin indices.

Now, the affine connection for spin in general relativity is expressed in terms of the Ricci rotation coefficients by $\Gamma_\mu = \frac{1}{8}\gamma_{ij\mu}(\gamma^i\gamma^j - \gamma^j\gamma^i) + a_\mu I$, where the γ^i are the Dirac matrices of special relativity, I is the identity matrix, and a_μ is an arbitrary vector. It is well known that the spin connection contains complete information about the electromagnetic field, and that one half of Maxwell's equations are identically satisfied on account of the existence of the spin connection. Furthermore, the manner in which the electromagnetic field enters the spin connection is in agreement with the principle of minimal electromagnetic coupling. An understanding of the spinor calculus in Riemann space, and the role played by the spin connection, was gained through the work of many investigators during the decade after Dirac's discovery of the relativistic theory of the electron; see e.g., Bade and Jehle (1953) for a general review. Many of these investigators recognized the description of the electromagnetic field as part of the spin connection. An especially lucid discussion of this has been given by Loos (1963). The subsequent unification of the electromagnetic and weak fields by Weinberg (1967), and Salam (1968) causes us to expect that the spin connection might also contain a description of the weak field.

We now recall (Pandres, 1995) calculations that suggest that the electroweak field is described by $M_{\mu\nu i}$, the "mixed symmetry" part of $\gamma_{\mu\nu i}$ under the permutation group on three symbols. One may object to using $\gamma_{\mu\nu i}$ to describe the electoweak field since $\gamma_{ij\mu}$ is used in the spin connection. However, these geometric objects cannot be considered to be the same since the method of converting from one to the other is not just a diffeomorphism but involves h^i_μ. Thus the components of $\gamma_{\mu\nu i}$ are independent of the components of $\gamma_{ij\mu}$, although if one is zero the other is also zero. The totally symmetric part of $\gamma_{\mu\nu i}$ vanishes because $\gamma_{\mu\nu i}$ is antisymmetric in μ and ν. Thus, we have $\gamma_{\mu\nu i} = M_{\mu\nu i} + A_{\mu\nu i}$, where $A_{\mu\nu i}$ is the totally antisymmetric part. Clearly, $A^\alpha{}_{\mu\nu}$ makes no contribution to the right side of Eq. (5), so

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{dx^\mu}{ds}M^\alpha{}_{\mu i}v^i, \tag{6}$$

where $v^i = \frac{dx^i}{ds}$ is the (constant) first integral of Eq. (2). The totally antisymmetric part of $\gamma_{\mu\nu i}$ is

$$A_{\mu\nu i} = \frac{1}{3}(\gamma_{\mu\nu i} + \gamma_{i\mu\nu} + \gamma_{\nu i\mu}). \tag{7}$$

Thus, the mixed symmetry part is $M_{\mu\nu i} = \gamma_{\mu\nu i} - A_{\mu\nu i}$, so, we have

$$M_{\mu\nu i} = \frac{1}{3}(2\gamma_{\mu\nu i} - \gamma_{i\mu\nu} - \gamma_{\nu i\mu}). \tag{8}$$

Define $f_{i\mu\nu} := h_{i\nu,\mu} - h_{i\mu,\nu} = h_{i\nu;\mu} - h_{i\mu;\nu}$, so that $f_{i\mu\nu} = \gamma_{i\nu\mu} - \gamma_{i\mu\nu}$. The antisymmetry of $\gamma_{\mu\nu i}$ in its first two indices may be used to obtain an expression

for $M_{\mu\nu i}$ in terms of $f_{i\mu\nu}$. If we subtract from this the corresponding expressions for $f_{\mu\nu i}$ and $f_{\nu i\mu}$, we see that $\gamma_{\mu\nu i} = \frac{1}{2}(f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu})$. By using this and the corresponding expressions for $\gamma_{i\mu\nu}$, and $\gamma_{\nu i\mu}$ in Eq. (8), we obtain

$$M_{\mu\nu i} = \frac{1}{3}(2 f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu}), \tag{9}$$

which may be written

$$M_{\mu\nu i} = \frac{1}{3}\left(2\delta_i^n \delta_\mu^\alpha \delta_\nu^\sigma - h_\mu^n \delta_\nu^\alpha h_i^\sigma - h_\nu^n h_i^\alpha \delta_\mu^\sigma\right) f_{n\alpha\sigma}, \tag{10}$$

where δ_μ^α is the Kronecker delta. It is important to notice that Eq. (10) may be rewritten into the form

$$M_{\mu\nu i} = \frac{1}{3}\left(2\delta_i^n \delta_\mu^\alpha \delta_\nu^\sigma - h_\mu^n \delta_\nu^\alpha h_i^\sigma - h_\nu^n h_i^\alpha \delta_\mu^\sigma\right) \mathfrak{F}_{n\alpha\sigma}, \tag{11}$$

where

$$\mathfrak{F}_{i\mu\nu} = f_{i\mu\nu} + e_{0ijk} h_\mu^j h_\nu^k, \tag{12}$$

and e_{nijk} is the Levi–Civita symbol. In rewriting Eq. (10) as Eq. (11), we have used the easily verifiable fact that

$$\left(2\delta_i^n \delta_\mu^\alpha \delta_\nu^\sigma - h_\mu^n \delta_\nu^\alpha h_i^\sigma - h_\nu^n h_i^\alpha \delta_\mu^\sigma\right) e_{0njk} h_\alpha^j h_\sigma^k = 0.$$

Now, $\mathfrak{F}_{i\mu\nu}$ is the usual field strength (see e.g., Nakahara, 1990) for a $U(1) \times SU(2)$ gauge field, *provided that h_μ^i is transformed on its tetrad indices as a gauge potential*, rather than as a Lorentz vector. We wish to make it clear that we will not require that h_μ^i be transformed as a gauge potential. In our view, the need for such a transformation rule arises from the fact that coordinate transformations are limited to the diffeomorphisms. In Section 2, we enlarge the group of diffeomorphisms to the conservation group. The mass-changing effect of a nondiffeomorphic conservative transformation is similar to what one would get if h_μ^i were to be transformed as a gauge potential. It is eminently reasonable that when a particle is subjected to a rotation in isospace the gravitational field may change.

From Eq. (11), we see that in the expression, Eq. (10), for $M_{\mu\nu i}$, the curl $f_{n\alpha\gamma}$ may simply be replaced by the gauge field $\mathfrak{F}_{n\alpha\gamma}$. The $\mathfrak{F}_{n\alpha\gamma}$ may be viewed as a field with "bare" or massless quanta, which are "clothed" by the factor $\frac{1}{3}(2\delta_i^n \delta_\mu^\alpha \delta_\nu^\sigma - h_\mu^n \delta_\nu^\alpha h_i^\sigma - h_\nu^n h_i^\alpha \delta_\mu^\sigma)$, and thus may acquire mass. The analysis in Section 2.4 suggests that $M_{\mu\nu i}$ may describe the physical electroweak field as it appears in the appropriate way in our Lagrangian, and in the stress-energy tensor of the Einstein equations. For this identification to be valid, the quantity $M_{\mu\nu 0} = \frac{1}{3}(2 f_{0\mu\nu} - f_{\mu\nu 0} - f_{\nu 0\mu})$ must describe the electromagnetic field; hence, it must be the curl of a vector. The presence of the terms $- f_{\mu\nu 0} - f_{\nu 0\mu}$ may cause one to ask how $M_{\mu\nu i}$ can be identified as the electroweak field.

Our answer is this: The orthodox physical interpretation, which we adopt, is that h^i_μ describes an observer-frame. Now, if h^i_μ *describes a freely falling, nonrotating observer frame, our expression for* $M_{\mu\nu 0}$ *reduces to* $M_{\mu\nu 0} = \frac{1}{3} f_{0\mu\nu}$. This may be seen as follows. The condition for a freely falling, nonrotating frame (Synge, 1960) is $h_{i\nu;\alpha} h^\alpha_0 = 0$. In terms of the Ricci rotation coefficients, the condition is $\gamma_{\mu\nu 0} = 0$. From this and Eq. (8), we see that for an h^i_μ which describes a freely falling, nonrotating observer frame, $M_{\mu\nu 0} = \frac{1}{3}(\gamma_{0\nu\mu} - \gamma_{0\mu\nu}) = \frac{1}{3}(h_{0\nu;\mu} - h_{0\mu;\nu}) = \frac{1}{3}(h_{0\nu,\mu} - h_{0\mu,\nu}) = \frac{1}{3} f_{0\mu\nu}$. Moreover, in the nonrelativistic limit (i.e., for v^1, v^2, v^2 small compared to one), the electromagnetic term $\frac{dx^\mu}{ds} M_\mu{}^\alpha{}_0 v^0$ dominates the right side of Eq. (6).

2. GRAVITATIONAL AND ELECTROWEAK UNIFICATION

It is clear that no meaningful physics can be done without an observer. Thus the principle of parsimony (Occam's razor) suggests that we consider a theory in which the observer-frame h^i_μ is the only fundamental field; i.e., in which geometrical structure is expressed by h^i_μ, rather than by $g_{\mu\nu}$. For this purpose we need an invariant Lagrangian constructed from h^i_μ and its derivatives, to be used in a variational principle (analogous to the Hilbert variational principle for gravitation, but with h^i_μ varied rather than $g_{\mu\nu}$).

2.1. Weitzenböck Invariants

Soon after Einstein (1928a,b) introduced tetrads into physics, Weitzenböck (1928) considered quantities constructed from h^i_μ and its derivatives which are invariant under the diffeomorphisms (the coordinate transformations of general relativity). Weitzenböck listed the invariants

$$A = \frac{1}{4} f^{i\mu\nu} f_{\nu\mu i} \quad B = \frac{1}{4} f^{i\mu\nu} f_{i\mu\nu} \quad \Phi = \frac{1}{4} C^\nu C_\nu \quad \Psi = \frac{1}{2}(C^\nu_{,\nu} + C^\mu h^\nu_i h^i_{\mu,\nu})$$

where the vector C_ν is defined by

$$C_\nu = h^\mu_i f^i{}_{\mu\nu}. \tag{13}$$

As Weitzenböck noted, the most general Lagrangian which yields second-order field equations that are linear in the second derivatives of h^i_μ is $L = aA + bB + \phi\Phi + \psi\Psi$, where the coefficients a, b, ϕ, ψ are constants.

In order to optimize clarity in our discussion, we shall use an equivalent list of invariants

$$W_1 = f^{i\mu\nu} f_{\nu\mu i} \quad W_2 = f^{i\mu\nu} f_{i\mu\nu} \quad W_3 = C^\nu C_\nu \quad W_4 = C^\nu_{;\nu}. \tag{14}$$

That our list is equivalent to Weitzenböck's is clear, because $W_1 = 4A$, $W_2 = 4B$, $W_3 = 4\Phi$, and $W_4 = C^{\nu}_{,\nu} + C^{\mu}\Gamma^{\nu}{}_{\mu\nu} = (C^{\nu}_{,\nu} + C^{\mu}h^{\nu}_{i}h^{i}{}_{\mu,\nu}) - C^{\mu}(h^{\nu}_{i}h^{i}{}_{\mu,\nu} - \Gamma^{\nu}{}_{\mu\nu}) = 2\Psi - C^{\mu}(h^{\nu}_{i}h^{i}h^{i}{}_{\mu,\nu} - \Gamma^{\nu}{}_{\mu\nu})$. We see from the definition of $\Gamma^{\alpha}{}_{\mu\nu}$ and Eq. (1) that $\Gamma^{\nu}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\nu}g_{\sigma\nu,\mu} = h^{\nu}_{i}h^{i}{}_{\nu,\mu}$. Thus, we have $W_4 = 2\Psi - C^{\mu}h^{\nu}_{i}(h^{i}{}_{\mu,\nu} - h^{i}{}_{\nu,\mu}) = 2\Psi - C^{\mu}h^{\nu}_{i}f^{i}{}_{\nu\mu} = 2\Psi - C^{\mu}C_{\mu} = 2\Psi - 4\Phi$.

Clearly, we may write the Lagrangian as

$$L = k_1W_1 + k_2W_2 + k_3W_3 + k_4W_4 \tag{15}$$

where the coefficients k_1, k_2, k_3, k_4, are constants.

Weitzenböck recognized that if the fields to be varied are just the components of $g_{\mu\nu}$, then there is essentially no freedom of choice for the coefficients. He showed that except for a common multiplicative constant, one must choose $a = -2$, $b = -1$, $\phi = -4$, $\psi = 4$; i.e., $k_1 = -\frac{1}{2}$, $k_2 = -\frac{1}{4}$, $k_3 = 1$, $k_4 = 2$. With this choice, L is just the Ricci scalar R, which is the Lagrangian for the free gravitational field. However, since the fields to be varied in our theory are the components of h^{i}_{μ}, there exists a nondenumerable infinity of inequivalent Lagrangians corresponding to different ratios of the constants k_1, k_2, k_3, k_4. Thus, we are confronted with a dilemma that was anticipated by Einstein (1949). He noted that with the introduction of a richer structure (such as our tetrad), the diffeomorphism group "will no longer determine the equations as strongly as in the case of the symmetric tensor as structure." Einstein also suggested the solution for this dilemma: "Therefore it would be most beautiful, if one were to succeed in expanding the group once more, analogous to the step which led from special relativity to general relativity." Einstein's suggestion was in accord with the prophetic statement by Dirac (1930) that "The growth of the use of transformation theory, as applied first to relativity and later to the quantum theory is the essence of the new method in theoretical physics. Further progress lies in the direction of making our equations invariant under wider and still wider transformations." Dirac went on to remark "This state of affairs is very satisfactory from a philosophical point of view, as implying an increasing recognition of the part played by the observer in himself introducing the regularities that appear in his observations. . . ." Dirac's remark supports our use of the observer-frame h^{i}_{μ} as the only fundamental field.

In Section 2.3, we shall see that $k_1W_1 + k_2W_2$ is not invariant under a group larger than the diffeomorphisms for any choice of the constants k_1 and k_2. By contrast, $k_3W_3 + k_4W_4$ is invariant under a group larger than the diffeomorphisms for arbitrary choice of k_3 and k_4. But, $W_4 = C^{\nu}_{,\nu}$ is a covariant divergence; so, the term k_4W_4 would make no contribution to field equations. Hence, we shall choose for our Lagrangian the invariant $W_3 = C^{\nu}C_{\nu}$. We shall see that this Lagrangian is just the sum of the gravitational Lagrangian R and terms which we tentatively label as the electroweak Lagrangian E. The terms R and E are each invariant only under the diffeomorphisms; it is their sum that is invariant under the larger group. By using Eq. (12), we also note that C_{ν} may be rewritten into the form $C_{\nu} = h^{iu}\mathfrak{F}_{i\mu\nu}$.

Thus, in the expression for C_μ, just as in the expression for $M_{\mu\nu i}$, the curl $f_{i\mu\nu}$ may simply be replaced by the gauge field $\mathfrak{F}_{i\mu\nu}$.

2.2. Holonomic and Anholonomic Coordinates

It is possible to establish a one-to-one correspondence between points x of the manifold and coordinates x^α (at least in finite coordinate patches). Such coordinates are called (Schouten, 1954) *holonomic* coordinates. Let transformation coefficients $X^{\tilde{\alpha}}_{\mu}$ have a nonzero determinant, and let the components of $X^{\tilde{\alpha}}_{\mu}$ have definite values at each point x. Then, these components are one-valued functions of holonomic coordinates, i.e., $X^{\tilde{\alpha}}_{\mu} = X^{\tilde{\alpha}}_{\mu}(x^\sigma)$. The relation

$$dx^{\tilde{\alpha}} = X^{\tilde{\alpha}}_{\mu}(x^\sigma)\, dx^\mu \tag{16}$$

establishes a one-to-one correspondence between coordinate increments dx^α and $dx^{\tilde{\alpha}}$. The inverse relation to Eq. (16) is

$$dx^\mu = X^{\mu}_{\tilde{\alpha}}(x^\sigma)\, dx^{\tilde{\alpha}} \tag{17}$$

where $X^{\mu}_{\alpha}(x^\sigma)$ is defined by $X^{\mu}_{\tilde{\alpha}} X^{\tilde{\alpha}}_{\nu} = \delta^{\mu}_{\nu}$. Eq. (16) may be integrated to give a one-to-one correspondence between coordinates x^μ and $x^{\tilde{\alpha}}$ if and only if

$$X^{\tilde{\alpha}}{}_{\nu,\mu} - X^{\tilde{\alpha}}{}_{\mu,\nu} = 0. \tag{18}$$

Thus, if Eq. (18) is satisfied, the $x^{\tilde{\alpha}}$ are also holonomic coordinates. If Eq. (18) is *not* satisfied, then the $x^{\tilde{\alpha}}$ are called (Schouten, 1954) *anholonomic coordinates*.

There does not exist a one-to-one correspondence between points x of the manifold and anholonomic coordinates. Thus, in an equation such as Eq. (17), the holonomic coordinates x^σ cannot be eliminated in favor of anholonomic coordinates $x^{\tilde{\alpha}}$. A transformation to anholonomic coordinates must be accompanied by what Schouten calls a "mitschleppen," i.e., a "dragging along" of the holonomic coordinates. (In this sense, holonomic and anholonomic coordinates are not on the same footing. They can be put on the same footing through the introduction of a path space, as we have done in several prior papers. In this paper, however, our setting is a manifold.) We can enlarge the covariance group so that it includes transformations to anholonomic coordinates, because our group elements are the transformation coefficients (which have definite values at each point x).

We shall need partial derivatives with respect to anholonomic as well as holonomic coordinates. Let F be a function with a definite value at each point x. If x^α and $x^{\tilde{\alpha}}$ are both holonomic, the relation between $F_{,\alpha}$ and $F_{,\tilde{\alpha}}$ is

$$F_{,\tilde{\alpha}} = F_{,\mu} X^{\mu}_{\tilde{\alpha}}. \tag{19}$$

Thus, regardless whether $x^{\tilde{\alpha}}$ is holonomic or anholonomic, we may take Eq. (19) as the *definition* of $F_{,\tilde{\alpha}}$ (where x^α remains holonomic). Let the coordinates $x^{\hat{\alpha}}$ also

be either holonomic or anholonomic. Then, of course, $F_{,\hat{\alpha}} = F_{,\mu} X^{\mu}_{\hat{\alpha}}$, and we easily find that $F_{,\hat{\alpha}} = F_{,\tilde{\mu}} X^{\tilde{\mu}}{}_{\hat{\alpha}}$, where $X^{\tilde{\mu}}{}_{\hat{\alpha}} = X^{\tilde{\mu}}_{\sigma} X^{\sigma}_{\hat{\alpha}}$.

2.3. The Conservation Group

The transformation law for a tetrad of vectors is

$$h^{i}_{\mu} = h^{i}_{\tilde{\alpha}} X^{\tilde{\alpha}}_{\mu}. \tag{20}$$

Upon differentiating Eq. (20) with respect to x^{ν}, we have

$$h^{i}{}_{\mu,\nu} = h^{i}{}_{\tilde{\alpha},\nu} X^{\tilde{\alpha}}_{\mu} + h^{i}_{\tilde{\alpha}} X^{\tilde{\alpha}}{}_{\mu,\nu} = h^{i}{}_{\tilde{\alpha},\tilde{\sigma}} X^{\tilde{\sigma}}_{\nu} X^{\tilde{\alpha}}_{\mu} + h^{i}_{\tilde{\alpha}} X^{\tilde{\alpha}}{}_{\mu,\nu}$$

If we subtract this from the corresponding expression with μ and ν interchanged, we obtain

$$f^{i}{}_{\mu\nu} = f^{i}{}_{\hat{\alpha}\hat{\sigma}} X^{\tilde{\alpha}}_{\mu} X^{\tilde{\sigma}}_{\nu} + h^{i}_{\tilde{\alpha}} \left(X^{\tilde{\alpha}}{}_{\nu,\mu} - X^{\tilde{\alpha}}{}_{\mu,\nu} \right) \tag{21}$$

where $f^{i}{}_{\hat{\alpha}\hat{\sigma}} = h^{i}{}_{\tilde{\sigma},\tilde{\alpha}} - h^{i}{}_{\tilde{\alpha},\tilde{\sigma}}$. We see from Eq. (21) that $f^{i}{}_{\mu\nu}$ transforms as a tensor if and only if Eq. (18) is satisfied, i.e., if and only if the transformation is a diffeomorphism. We also see from Eqs. (14) and (21) that no linear combination of W_1 and W_2 with constant coefficients is invariant under a larger group than the diffeomorphisms. By contrast, if we multiply Eq. (21) by $h^{\mu}_{i} = h^{\tilde{\rho}}_{i} X^{\mu}_{\tilde{\rho}}$ and use Eq. (13), we get

$$C_{\nu} = C_{\tilde{\alpha}} X^{\tilde{\alpha}}_{\nu} + X^{\mu}_{\tilde{\alpha}} \left(X^{\tilde{\alpha}}{}_{\nu,\mu} - X^{\tilde{\alpha}}{}_{\mu,\nu} \right). \tag{22}$$

We see from Eq. (22) that C_{ν} transforms as a vector if and only if

$$X^{\nu}_{\tilde{\alpha}} \left(X^{\tilde{\alpha}}{}_{\nu,\mu} - X^{\tilde{\alpha}}{}_{\mu,\nu} \right) = 0. \tag{23}$$

Accordingly, we recall (Pandres, 1981) that $C^{\nu} C_{\nu}$ is invariant under transformations that satisfy Eq. (23).

2.3.1. Conservative Coordinate Transformations

In the discussion that led to Eq. (23), x^{α} was required to be holonomic. We now relax that requirement and allow x^{α} and/or $x^{\tilde{\alpha}}$ to be either holonomic or anholonomic. A transformation which satisfies Eq. (23) is called *conservative*. This terminology is appropriate for the following reason: A relativistic conservation law is an expression of the form $V^{\alpha}_{,\alpha} = 0$, where V^{α} is a vector density of weight $+1$. This is a covariant statement under a coordinate transformation relating x^{α} and $x^{\tilde{\alpha}}$ if and only if it implies and is implied by the relation $V^{\tilde{\alpha}}_{,\tilde{\alpha}} = 0$. The transformation law for a vector density of weight $+1$ is $V^{\tilde{\alpha}} = \frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}_{\mu} V^{\mu}$, where $\frac{\partial x}{\partial \tilde{x}}$ is the (nonzero) Jacobian determinant of $X^{\mu}_{\tilde{\alpha}}$. Upon differentiating $V^{\tilde{\alpha}}$ with respect to $x^{\tilde{\alpha}}$, we obtain

$V^{\tilde{\alpha}}_{,\tilde{\alpha}} = \left(\frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}_{\mu}\right)_{,\hat{\alpha}} V^{\mu} + \frac{\partial x}{\partial \tilde{x}} V^{\alpha}_{,\alpha}$. For arbitrary V^{μ}, we see that a conservation law is a covariant statement if and only if

$$\left(\frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}_{\mu}\right)_{,\tilde{\alpha}} = 0. \tag{24}$$

For this reason, we call a coordinate transformation conservative if it satisfies Eq. (24). Now,

$$\left(\frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}_{\mu}\right)_{,\tilde{\alpha}} = \left(\frac{\partial x}{\partial \tilde{x}}\right)_{,\tilde{\alpha}} X^{\tilde{\alpha}}_{\mu} + \frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}{}_{\mu,\tilde{\alpha}} = \left(\frac{\partial x}{\partial \tilde{x}}\right)_{,\mu} + \frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}{}_{\mu,\nu} X^{\nu}_{\tilde{\alpha}}$$

so, if we use the well-known formula

$$\left(\frac{\partial x}{\partial \tilde{x}}\right)_{,\mu} = \frac{\partial x}{\partial \tilde{x}} X^{\tilde{\alpha}}_{\nu} X^{\nu}{}_{,\tilde{\alpha},\mu}$$

for the derivative of a determinant, and note that $X^{\tilde{\alpha}}_{\nu} X^{\nu}{}_{\tilde{\alpha},\mu} = -X^{\tilde{\alpha}}{}_{\nu,\mu} X^{\nu}_{\tilde{\alpha}}$, we find that Eq. (24) is equivalent to Eq. (23).

2.3.2. The Conservation Group

We now recall (Pandres, 1981) an explicit proof that the conservative coordinate transformations form a group. [D. Finkelstein (private communication, 1981), however, has pointed out that the group property follows implicitly from the derivation given above.] First, we note that the identity transformation $x^{\tilde{\alpha}} = x^{\alpha}$ is a conservative coordinate transformation. Next, we consider the result of following a coordinate transformation from x^{α} to $x^{\tilde{\alpha}}$ by a coordinate transformation from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$. Upon differentiating

$$X^{\hat{\alpha}}_{\mu} = X^{\hat{\alpha}}_{\tilde{\rho}} X^{\tilde{\rho}}_{\mu} \tag{25}$$

with respect to x^{ν}, subtracting the corresponding expression with μ and ν interchanged, and multiplying by $X^{\nu}_{\tilde{\alpha}}$ we obtain

$$X^{\nu}_{\tilde{\alpha}}\left(X^{\hat{\alpha}}{}_{\nu,\mu} - X^{\hat{\alpha}}{}_{\mu,\nu}\right) = X^{\tilde{\rho}}_{\mu} X^{\tilde{\sigma}}_{\hat{\alpha}}\left(X^{\hat{\alpha}}{}_{\tilde{\sigma},\tilde{\rho}} - X^{\hat{\alpha}}{}_{\tilde{\rho},\tilde{\sigma}}\right) + X^{\nu}_{\tilde{\rho}}\left(X^{\hat{\rho}}{}_{\nu,\mu} - X^{\hat{\rho}}{}_{\mu,\nu}\right). \tag{26}$$

We see from Eq. (26) that if $X^{\nu}_{\tilde{\rho}}(X^{\hat{\rho}}{}_{\nu,\mu} - X^{\hat{\rho}}{}_{\mu,\nu})$ and $X^{\tilde{\sigma}}_{\hat{\alpha}}(X^{\hat{\alpha}}{}_{\tilde{\sigma},\tilde{\rho}} - X^{\hat{\alpha}}{}_{\tilde{\rho},\tilde{\sigma}})$ vanish, then $X^{\nu}_{\tilde{\alpha}}(X^{\hat{\alpha}}{}_{\nu,\mu} - X^{\hat{\alpha}}{}_{\mu,\nu})$ vanishes. This shows that if the transformations from x^{α} to $x^{\tilde{\alpha}}$ and from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$ are conservative coordinate transformations, then the product transformation from x^{α} to $x^{\hat{\alpha}}$ is a conservative coordinate transformation. If we let $x^{\hat{\alpha}} = x^{\alpha}$, we see from Eq. (26) that the inverse of a conservative coordinate transformation is a conservative coordinate transformation. From Eq. (25), we see that the product of matrices $X^{\tilde{\rho}}_{\mu}$ and $X^{\hat{\alpha}}_{\tilde{\rho}}$ (which represent the transformations from x^{α} to $x^{\tilde{\alpha}}$ and from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$, respectively) equals the matrix $X^{\hat{\alpha}}_{\mu}$ (which represents the product transformation from x^{α} to $x^{\hat{\alpha}}$). It is obvious, and well known, that if

products admit a matrix representation in this sense, then the associative law is satisfied. This completes the proof that the conservative coordinate transformations form a group, which we call the conservation group.

To show that the conservation group contains the diffeomorphisms as a proper subgroup, we need only exhibit transformation coefficients which satisfy Eq. (23), but do not satisfy Eq. (18). Let

$$X_{\mu}^{\tilde{\alpha}} = \delta_{\nu}^{\tilde{\alpha}} + \delta_{0}^{\tilde{\alpha}}\delta_{\nu}^{2}x^{1}. \tag{27}$$

Upon differentiating Eq. (27) with respect to x^{μ} and subtracting the corresponding expression with μ and ν interchanged, we obtain

$$X^{\tilde{\alpha}}{}_{\nu,\mu} - X^{\tilde{\alpha}}{}_{\mu,\nu} = \delta_{0}^{\alpha}\left(\delta_{\mu}^{1}\delta_{\nu}^{2} - \delta_{\nu}^{1}\delta_{\mu}^{2}\right). \tag{28}$$

A nonzero component of Eq. (28) is $X_{2,1}^{\tilde{0}} - X_{1,2}^{\tilde{0}} = 1$, which shows that Eq. (18) is not satisfied. It is easily verified that

$$X_{\tilde{\alpha}}^{\nu} = \delta_{\tilde{\alpha}}^{\nu} - \delta_{0}^{\nu}\delta_{\tilde{\alpha}}^{2}x^{1} \tag{29}$$

satisfies our condition $X_{\tilde{\alpha}}^{\mu}X_{\nu}^{\tilde{\alpha}} = \delta_{\nu}^{\mu}$. If we multiply Eq. (28) by Eq. (29), we see that Eq. (23) is satisfied.

2.4. The Lagrangian

We now recall (Pandres, 1999) evidence that the invariant $W_3 = C^{\nu}C_{\nu}$ is an appropriate Lagrangian for gravitational and electroweak unification.

The Riemann tensor is defined as usual by $R^{\alpha}{}_{\beta\mu\nu} = h_{i}^{\alpha}(h^{i}{}_{\beta;\mu;\nu} - h^{i}{}_{\beta;\nu;\mu})$ while the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined, as usual, by $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = R_{\alpha}^{\alpha}$. By using $h_{i}^{\alpha}h^{i}{}_{\beta;\mu;\nu} = (h_{i}^{\alpha}h^{i}{}_{\beta;\mu})_{;\nu} - h^{\alpha}{}_{i;\nu}h^{i}{}_{\beta;\mu} = \gamma^{\alpha}{}_{\beta\mu;\nu} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\beta\mu}$, we easily find that

$$R^{\alpha}{}_{\beta\mu\nu} = \gamma^{\alpha}{}_{\beta\mu;\nu} - \gamma^{\alpha}{}_{\beta\nu;\mu} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\beta\mu} - \gamma^{\alpha}{}_{\sigma\mu}\gamma^{\sigma}{}_{\beta\nu}. \tag{30}$$

From Eq. (13), we see that $C_{\mu} = h_{i}^{\nu}(h^{i}{}_{\mu,\nu} - h^{i}{}_{\nu,\mu}) = h_{i}^{\nu}(h_{\mu;\nu}^{i} - h_{\nu;\mu}^{i}) = \gamma^{\nu}{}_{\mu\nu} - \gamma^{\nu}{}_{\nu\mu} = \gamma^{\nu}{}_{\mu\nu}$. By using $C_{\mu} = \gamma^{\nu}{}_{\mu\nu}$, we find from Eq. (30) that

$$R_{\mu\nu} = C_{\mu;\nu} - C_{\alpha}\gamma^{\alpha}{}_{\mu\nu} - \gamma^{\alpha}{}_{\mu\nu;\alpha} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\mu\alpha}, \tag{31}$$

and, from Eq. (31)

$$C^{\mu}C_{\mu} = R + \gamma^{\mu i\nu}\gamma_{\mu\nu i} - 2C_{;\mu}^{\mu}. \tag{32}$$

The first term on the right side of Eq. (32) is the Ricci scalar, which is the Lagrangian for gravitation. The last term is a covariant divergence, which contributes nothing to the field equations. We now consider the interpretation of the term $\gamma^{\mu i\nu}\gamma_{\mu\nu i}$.

From Eqs. (7) and (8), we see that

$$A^{\mu\nu i} M_{\mu\nu i} = 0, \tag{33}$$

and that

$$M_{\mu\nu i} + M_{i\mu\nu} + M_{\nu i\mu} = 0. \tag{34}$$

From $\gamma_{\mu\nu i} = M_{\mu\nu i} + A_{\mu\nu i}$, and Eq. (33), we get $\gamma^{\mu i\nu}\gamma_{\mu\nu i} = M^{\mu i\nu} M_{\mu\nu i} - A^{\mu\nu i} A_{\mu\nu i}$. But, $M^{\mu i\nu} M_{\mu\nu i} = \frac{1}{2} M^{\mu i\nu} M_{\mu\nu i} + \frac{1}{2} M^{\nu i\mu} M_{\nu\mu i} = \frac{1}{2} M^{\mu i\nu} M_{\mu\nu i} + \frac{1}{2} M^{i\nu\mu} M_{\mu\nu i} = \frac{1}{2} \left(M^{\mu i\nu} + M^{i\nu\mu} \right) M_{\mu\nu i} = \frac{1}{2} M^{\mu\nu i} M_{\mu\nu i}$, where we have used Eq. (34). Thus, we have $\gamma^{\mu i\nu}\gamma_{\mu\nu i} = \frac{1}{2} M^{\mu\nu i} M_{\mu\nu i} - A^{\mu\nu\alpha} A_{\mu\nu\alpha}$. We now define a vector

$$A^{\mu} = \frac{1}{3!} (-g)^{-1/2} e^{\mu\alpha\beta\sigma} A_{\alpha\beta\sigma}, \tag{35}$$

and find that

$$A^{\mu\nu\alpha} A_{\mu\nu\alpha} = -6 A^{\mu} A_{\mu}. \tag{36}$$

In obtaining Eq. (36), we have used the well known identity (see e.g., Weber, 1961) for expressing the product of two Levi–Civita symbols as a determinant of Kronecker deltas. We now see that Eq. (32) may be written

$$C^{\mu} C_{\mu} = R + \frac{1}{2} M^{\mu\nu i} M_{\mu\nu i} + 6 A^{\mu} A_{\mu} - 2 C^{\mu}_{;\mu}. \tag{37}$$

The term $M^{\mu\nu i} M_{\mu\nu i}$ is in the form of the usual electroweak Lagrangian, and the $A^{\mu} A_{\mu}$ term has precisely the form that is needed (see e.g., Moriyasu, 1983) for the introduction of mass.

2.5. Field Equations

We have previously (Pandres, 1981) considered the variational principle

$$\delta \int C^{\mu} C_{\mu} \sqrt{-g}\, d^4x = 0$$

where h^i_{μ} is varied. We note that $\sqrt{-g}$ equals h, the determinant of h^i_{μ}; and that $C^{\mu} C_{\mu} = C^i C_i$. Hence, our variational principle may be written

$$\delta \int C^i C_i h\, d^4x = 0. \tag{38}$$

The variational calculation (Pandres, 1984a) using $C^i C_i$ is less tedious than that using $C^{\mu} C_{\mu}$. We find from Eq. (38) that

$$\int h \left(2 C^i \delta C_i - C^i C_i h^k_{\nu} \delta h^{\nu}_k \right) d^4x = 0, \tag{39}$$

where we have used $\delta h = h h^{\nu}_{k} \delta h^{k}_{\nu} = -h h^{k}_{\nu} \delta h^{\nu}_{k}$. We note that

$$\begin{aligned}\left(h h^{\nu}_{i}\right)_{,\nu} &= h_{,\nu} h^{\nu}_{i} + h h^{\nu}_{i,\nu} \\ &= h\left(h^{\mu}_{k} h^{k}{}_{\mu,\nu} h^{\nu}_{i} + h^{\nu}_{k,\nu} h^{k}_{\mu} h^{\mu}_{i}\right) \\ &= h\left(h^{\nu}_{k} h^{k}{}_{\nu,\mu} h^{\mu}_{i} - h^{\nu}_{k} h^{k}{}_{\mu,\nu} h^{\mu}_{i}\right) \\ &= -h C_{\mu} h^{\mu}_{i} = -h C_{i}\end{aligned}$$

Thus, we see that

$$C_{i} = -h^{-1}\left(h h^{\nu}_{i}\right)_{,\nu}. \tag{40}$$

Variation of Eq. (40) gives $\delta C_{i} = h^{-2}(h h^{\nu}_{i})_{,\nu}\delta h - h^{-1}\delta(h h^{\nu}_{i})_{,\nu} = C_{i} h^{k}_{\nu} \delta h^{\nu}_{k} - h^{-1}\left[\delta(h h^{\nu}_{i})\right]_{,\nu}$. Upon using this expression for δC_{i} in Eq. (39), we obtain

$$\int h C^{k} C_{k} h^{i}_{\nu} \delta h^{\nu}_{i}\, d^{4}x - 2\int C^{i}\left[\delta\left(h h^{\nu}_{i}\right)\right]_{,\nu} d^{4}x = 0, \tag{41}$$

and, integration by parts gives

$$\int h\left(C^{i}_{,\nu} - h^{i}_{\nu} C^{k}_{,k} + \frac{1}{2} h^{i}_{\nu} C^{k} C^{k}\right)\delta h^{\nu}_{i}\, d^{4}x - \int\left[C^{i}\delta\left(h h^{\nu}_{i}\right)\right]_{,\nu} d^{4}x = 0. \tag{42}$$

By using Gauss's theorem, we may write the second integral of Eq. (42) as an integral over the boundary of the region of integration. We discard this boundary integral by demanding that $C^{i}\delta(h h^{\nu}_{i})$ shall vanish on the boundary, and demand that δh^{ν}_{i} be arbitrary in the interior of the (arbitrary) region of integration. Thus, we get field equations $C^{i}_{,\nu} - h^{i}_{\nu} C^{k}_{,k} + \frac{1}{2} h^{i}_{\nu} C^{k} C_{k} = 0$, and, upon multiplying by h^{ν}_{j}, we write these field equations as

$$C^{i}_{,j} - \delta^{i}_{j} C^{k}_{,k} + \frac{1}{2}\delta^{ij} C^{k} C_{k} = 0. \tag{43}$$

We note that $C^{\alpha}_{;\sigma} = (C^{k} h^{\alpha}_{k})_{;\sigma} = C^{k}_{,\sigma} h^{\alpha}_{k} + C^{k} h^{\alpha}_{k;\sigma} = C^{k}_{,\sigma} h^{\alpha}_{k} + C^{k}\gamma_{k}{}^{\alpha}{}_{\sigma}$. Thus, we have $C^{k}_{,\sigma} h^{\alpha}_{k} = C^{\alpha}_{;\sigma} + C^{\rho}\gamma^{\alpha}_{\rho\sigma}$. If we multiply by $h^{i}_{\alpha} h^{\sigma}_{j}$, we get $C^{i}_{,j} = h^{i}_{\alpha} h^{\sigma}_{j}(C^{\alpha}_{;\sigma} + C^{\rho}\gamma^{\alpha}_{\rho\sigma})$, and $C^{k}_{,k} = C^{\alpha}_{;\alpha} + C^{\alpha} C_{\alpha}$. If we use these expressions for $C^{i}_{,j}$ and $C^{k}_{,k}$ in Eq. (43), we obtain the relation $h^{i}_{\alpha} h^{\sigma}_{j}(C^{\alpha}_{;\sigma} + C^{\rho}\gamma^{\alpha}{}_{\rho\sigma}) - \delta^{i}_{j} C^{\alpha}_{;\alpha} - \frac{1}{2}\delta^{i}_{j} C^{\alpha} C_{\alpha} = 0$, and, upon multiplying this by $h_{i\mu} h^{j}_{\nu}$, we rewrite our field equations as

$$C_{\mu;\nu} - C_{\alpha}\gamma^{\alpha}{}_{\mu\nu} - g_{\mu\nu} C^{\alpha}_{;\alpha} - \frac{1}{2} g_{\mu\nu} C^{\alpha} C_{\alpha} = 0. \tag{44}$$

2.5.1. The Field Equations as Einstein Equations

The Einstein equations of general relativity may be interpreted in two ways. One interpretation is as differential equations for the metric, when the stress-energy

tensor is given. Alternatively, these equations may be looked upon as a definition of the stress-energy tensor in terms of the metric. The second interpretation has been stressed particularly by Schrödinger (1960) ["I would rather you did not regard these equations as field equations, but as a definition of T_{ik} the matter tensor."] and by Eddington (1924) ["and we must proceed by inquiring first what experimental properties the physical tensor possesses, and then seeking a geometrical tensor which possesses these properties"]. It is the second interpretation that we adopt.

From Eqs. (31) and (32), we find that an identity for the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is

$$G_{\mu\nu} = C_{\mu;\nu} - C_\alpha \gamma^\alpha{}_{\mu\nu} - g_{\mu\nu}C^\alpha_{;\alpha} - \frac{1}{2}g_{\mu\nu}C^\alpha C_\alpha + \gamma_\mu{}^\alpha{}_{\nu;\alpha} + \gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\mu;\alpha} + \frac{1}{2}g_{\mu\nu}\gamma^{\alpha i\sigma}\gamma_{\alpha\sigma i}. \tag{45}$$

Equation (44) just states that the first line on the right side of Eq. (45) vanishes. Thus, we may write our field equations as

$$G_{\mu\nu} = \gamma^\alpha_{\mu\nu;\alpha} + \gamma^\alpha_{\sigma\nu}\gamma^\sigma_{\mu\alpha} + \frac{1}{2}g_{\mu\nu}\gamma^{\alpha i\sigma}\gamma_{\alpha\sigma i}. \tag{46}$$

By using the well known symmetry of the Einstein tensor, i.e., $G_{\mu\nu} = G_{\nu\mu}$. we see from Eq. (46) that the symmetric part of our field equations is

$$G_{\mu\nu} = \frac{1}{2}\left(\gamma_\mu{}^\alpha{}_\nu + \gamma_\nu{}^\alpha{}_\mu\right)_{;\alpha} + \frac{1}{2}\left(\gamma^\alpha{}_{\sigma\mu}\gamma^\sigma{}_{\nu\alpha} + \gamma^\alpha{}_{\sigma\mu}\gamma^\sigma{}_{\nu\alpha}\right) + \frac{1}{2}g_{\mu\nu}\gamma^{\alpha i\sigma}\gamma_{\alpha\sigma i}. \tag{47}$$

Since $\gamma^\alpha{}_{\mu\nu} = M^\alpha{}_{\mu\nu} + A^\alpha{}_{\mu\nu}$, we see that $(\gamma_\mu{}^\alpha{}_\nu + \gamma_\nu{}^\alpha{}_\mu)_{;\alpha} = (M^\alpha{}_{\mu\nu} + M_\nu{}^\alpha{}_\mu)_{;\alpha} = (M_\mu{}^\alpha{}_i h^i_\nu + M_\nu{}^\alpha{}_i h^i_\mu)_{;\alpha} = J_{\mu i}h^i_\nu + J_{\nu i}h^i_\mu + M_{\mu\sigma}{}^\alpha\gamma^\sigma{}_{\nu\alpha} + M_{\nu\sigma}{}^\alpha\gamma^\sigma{}_{\mu\alpha}$, where $J_{\mu i} = M^\alpha_{\mu i;\alpha}$ is a (conserved) electroweak current. From Eq. (33), the repeated use of Eq. (34), the total antisymmetry of $A_{\mu\nu\alpha}$, and the antisymmetries of $\gamma_{\mu\nu\alpha}$ and $M_{\mu\nu\alpha}$ in their first two indices, we find after a tedious but straightforward calculation that Eq. (47) may be written

$$G_{\mu\nu} = A^{ij}_\mu A_{ij\nu} - \frac{1}{2}g_{\mu\nu}A^{ij\alpha}A_{ij\alpha} + \frac{1}{2}\left(J_{\mu i}h^i_\nu + J_{\nu i}h^i_\mu\right) - M_{\mu\nu}, \tag{48}$$

where $M_{\mu\nu} = M^\alpha_{\mu i}M^i{}_{\alpha\nu} - \frac{1}{4}g_{\mu\nu}M^{\alpha\sigma i}M_{\alpha\sigma i}$. The terms in Eq. (48) that involve $A_{ij\mu}$ may be written in a more simple form. From Eq. (35), we have $A_\mu = \frac{1}{3!}(-g)^{-1/2}$
$g_{\mu\rho}\,e^{\rho\alpha\beta\sigma}A_{\alpha\beta\sigma}$, and we find after a little work that $A_\mu = -\frac{1}{3!}(-g)^{1/2}\,e_{\mu\alpha\beta\sigma}A^{\alpha\beta\sigma}$. Thus, $A_\mu A_\nu = -\frac{1}{36}g_{\mu\rho}\,e^{\rho\alpha\beta\sigma}\,e_{\nu\theta\lambda\tau}A^{\theta\lambda\tau}A_{a\beta\sigma}$. By expressing the product of Levi–Civita symbols as a determinant of Kronecker deltas, we get $A_\mu A_\nu = \frac{1}{2}A^{ij}{}_\mu A_{ij\nu} - \frac{1}{6}g_{\mu\nu}A^{ij\alpha}A_{ij\alpha}$. From this and Eq. (36), we see that Eq. (48) may be written

$$G_{\mu\nu} = 2A_\mu A_\nu + g_{\mu\nu}A^\alpha A_\alpha + \frac{1}{2}\left(J_{\mu i}h^i_\nu + J_{\nu i}h^i_\mu\right) - M_{\mu\nu}. \tag{49}$$

The right side of Eq. (49) is just what one would expect for the stress-energy tensor of the electroweak field, its associated currents, and gauge symmetry breaking terms corresponding to those in the Lagrangian, Eq. (37).

2.6. Solutions of the Field Equations

2.6.1. Solutions With $C_i = 0$

It is clear that our field equations, Eq. (43) are satisfied if $C_i = 0$. Consider the tetrad $h^i_\mu = \delta^i_\mu + \delta^i_0\delta^2_\mu x^1$, where x^1 is a Greek (space–time) coordinate. We have shown (Pandres, 1981) that this tetrad yields $C_i = 0$, gives a Ricci scalar $R = \frac{1}{2}$, and gives a metric $g_{\mu\nu}$ which satisfies the well known (Synge, 1960) Einstein equations for a charged dust cloud.

2.6.2. Solutions With C_i Constant and Lightlike

It is also clear that our field equations are satisfied if C_i is constant and lightlike. Consider the tetrad

$$h^i_\mu = \delta^i_\mu + \left(\delta^i_0 + \delta^i_1\right)\delta^0_\mu\left(e^{x_1} - 1\right) \tag{50}$$

where the coordinate x^1 is Greek. We have shown (Pandres, 1984a) that this tetrad yields a nonvanishing but constant and lightlike C_i.

2.6.3. Solutions With C_i which Does Not Vanish, and Is Neither Constant Nor Lightlike

It is clear from Sections 2.6.1 and 2.6.2 above that a tetrad satisfies our field equations if it satisfies the condition of either vanishing or being constant and lightlike. In a previous paper (Pandres, 1984a), we made the false assertion that a tetrad satisfies our field equations *only* if it satisfies this condition. The false assertion was based on the following argument: It is clear that for distinct values of i and j, the field equations state that $C^i_{,j} = 0$. This fact led us to assume that the component C^i can depend only on the single coordinate x^i; i.e., that $C^0_{,0}$ can depend only on x^0; $C^1_{,1}$ only on x^1, etc. This assumption would be true if the Latin coordinates were holonomic, but is false, because they are anholonomic. An example of a tetrad which satisfies our field equations, but yields a C_i which is neither constant nor lightlike has been found by one of us (Green). His tetrad is

$$h^\mu_i = \left[\left(x^0\right)^2 - \left(x^1\right)^2 + \left(x^2\right)^2 - \left(x^3\right)\right]\delta^\mu_i. \tag{51}$$

In Eq. (51), the coordinates x^0, x^1, x^2, x^3 are Greek. The tetrad in Eq. (51) satisfies our field equations, but yields $C_i = (-6x^0, -6x^1, -6x^2, -6x^3)$ which is neither constant nor lightlike.

2.6.4. Solutions That Yield Flat Riemann Space–Times

We note that our field equations admit nontrivial solutions for which $g_{\mu\nu}$ is the metric of a flat space–time. One of us (Green, 1991, reported in Pandres (1995)) has exhibited the tetrad

$$h_i^\mu = \delta_0^\mu \delta_i^0 + \delta_3^\mu \delta_i^3 + \left(\delta_1^\mu \delta_i^1 + \delta_2^\mu \delta_i^2\right) \cos x^3 + \left(\delta_2^\mu \delta_i^1 - \delta_1^\mu \delta_i^2\right) \sin x^3, \tag{52}$$

where the coordinate x^3 is Greek. For this h_i^μ, the quantity $M_{\mu\nu i}$ does not vanish, but $C_i = 0$, and $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. He has also exhibited (Green, 1997, reported in Pandres (1999)) the tetrad

$$h_i^\mu = \frac{1}{2}\left[\left(\delta_0^\mu \delta_i^0 + \delta_1^\mu \delta_i^1\right)\left(F + \frac{1}{F}\right) + \left(\delta_0^\mu \delta_i^1 + \delta_1^\mu \delta_i^0\right)\left(F - \frac{1}{F}\right)\right] + \delta_2^\mu \delta_i^2 + \delta_3^\mu \delta_i^3 \tag{53}$$

where $F = x^0 + x^1$, and the coordinates x^0 and x^1 are Greek. For this h_i^μ, the quantity $M_{\mu\nu i}$ does not vanish, but C_i is constant and lightlike, and $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

3. THE HAMILTONIAN

In this section the Hamiltonian for this theory will be derived. As the dynamics is constrained, the Dirac–Bergmann procedure will be used to find all the constraints and to produce a consistent Hamiltonian.

3.1. The Primary Hamiltonian

Let h be the determinant of h_μ^i as before. The 16 canonical position variables are defined by

$$Q_i^\alpha \equiv h h_i^\alpha. \tag{54}$$

Let Q_α^i be the matrix inverse of Q_i^α, so that $Q_i^\alpha Q_\beta^i = \delta_\beta^\alpha$ and $Q_\alpha^i Q_j^\alpha = \delta_j^i$. Let Q be the determinant of Q_i^α. Then $Q = \det(Q_i^\alpha) = \det(h h_i^\alpha) = h^4 h^{-1} = h^3$, and so, $h = Q^{\frac{1}{3}}$. Using Eq. (40) we have

$$C_i = -Q^{-\frac{1}{3}} Q^\nu_{i,\nu}, \tag{55}$$

and therefore the Lagrangian density may be expressed by

$$L = g^{ij} Q^\mu{}_{i,\mu} Q^\mu{}_{j,\nu} Q^{-\frac{1}{3}}. \tag{56}$$

The momenta are defined as usual by $P_\mu^i = \frac{\delta L}{\delta Q^\mu{}_{i,0}}$ where $Q^\mu{}_{i,0}$ is the derivative of Q_i^μ with respect to the Greek x^0 variable. We assume that the x^0 variable has a time-like direction at each point in space–time. We also assume that the values

of Q_i^α and P_μ^i as well as their derivatives on a space-like surface σ determine the dynamics. From Eq. (56) we have

$$P_\mu^i = 2g^{ij}Q^\nu{}_{j,\nu}\delta_\mu^0 Q^{-\frac{1}{3}}. \tag{57}$$

For the remainder of this section we will use a bar over an index to indicate a restriction of the index range to the values 1, 2, and 3. Thus there are 12 primary constraints

$$P_{\bar\mu}^i = 0 \quad \bar\mu = 1, 2, 3. \tag{58}$$

The four nonzero momenta are seen to be a multiple of the Latin components of the curvature vector:

$$P_0^i = 2g^{ij}Q^\nu{}_{j,\nu}Q^{-\frac{1}{3}} = -2g^{ij}C_j = -2C^i. \tag{59}$$

The Hamiltonian density H is defined by $H = P_\mu^i Q^\mu{}_{i,0} - L$. Using the constraints and Eq. (59), we have

$$H = P_0^i Q^0{}_{i,0} - \frac{1}{4}g_{ij}P_0^i P_0^j Q^{\frac{1}{3}}.$$

But using Eq. (55), $Q^0{}_{i,0} = Q^\mu{}_{i,\mu} - Q^{\bar\mu}{}_{i,\mu} = -Q^{\frac{1}{3}}C_i - Q^{\bar\mu}{}_{i,\bar\mu,} = \frac{1}{2}Q^{\frac{1}{3}}g_{ij}P_0^j - Q^{\bar\mu}{}_{i\mu}$ and therefore

$$\begin{aligned} H &= \frac{1}{2}g_{ij}P_0^i P_0^j Q^{\frac{1}{3}} - P_0^i Q^{\bar\mu}{}_{i,\bar\mu} - \frac{1}{4}g_{ij}P_0^i P_0^j Q^{\frac{1}{3}} \\ &= \frac{1}{4}g_{ij}P_0^i P_0^j Q^{\frac{1}{3}} - P_0^i Q^{\bar\mu}{}_{i,\bar\mu}. \end{aligned}$$

Hence we have the following primary Hamiltonian density

$$H_p = \frac{1}{4}g_{ij}P_0^i P_0^j Q^{\frac{1}{3}} - P_0^i Q^{\bar\mu}{}_{i,\bar\mu} + v_i^{\bar\mu}P_{\bar\mu}^i \tag{60}$$

with Lagrange multipliers $v_i^{\bar\mu}$. In cases where C^i is zero on the boundary of σ, a partial integration yields the following primary Hamiltonian density:

$$H_p = \frac{1}{4}g_{ij}P_0^i P_0^j Q^{\frac{1}{3}} + Q_i^{\bar\mu}P^i{}_{0,\bar\mu} + v_i^{\bar\mu}P_{\bar\mu}^i. \tag{61}$$

The full Hamiltonian is $\mathcal{H}_p = \int_\sigma H_p\, d^3x$.

3.2. The Dirac–Bergmann Procedure and the Consistent Hamiltonian

When the dynamics are constrained, consistency requires that the time derivative of a constraint must be zero. We follow the Dirac–Bergmann algorithm (Bergmann and Goldberg, 1955; Dirac, 1964; Sundermeyer, 1982; Weinberg,

1995) for constrained dynamics. The Hamiltonian equations of motion state that for any function X of Q_i^μ and P_μ^i we have

$$\frac{dX}{dt} = \{X, H_p\} = \frac{\delta X}{\delta Q_i^\mu}\frac{\delta H_p}{\delta P_\mu^i} - \frac{\delta X}{\delta P_\mu^i}\frac{\delta H_p}{\delta Q_i^\mu}.$$

Dirac refers to the constraint equations as weak equations, since one must be careful to use these equations only after time derivatives and other variations are calculated. We shall use x to represent the variable of integration for the first functional in the bracket and y to represent the variable of integration for the second functional. When there is no possible confusion these variables will be suppressed. We will also suppress display of the integrals over σ and only show the integrands.

Since, in general, C_i may have any value on the boundary, we proceed from Eq. (60). Substituting $W = g_{ij}P_0^iP_0^j$, and noting that $\frac{\delta Q}{\delta Q_i^\beta} = QQ_\beta^i$, we have

$$\frac{dP_{\bar\alpha}^i}{dt} = -\frac{\delta H_p}{\delta Q_i^{\bar\alpha}} = -\frac{1}{12}WQ^{\frac{1}{3}}Q_{\bar\alpha}^i\delta(x-y) + P_0^i\delta_{,\bar\alpha}(x-y)$$

and thus for consistency

$$\frac{1}{12}WQ^{\frac{1}{3}}Q_{\bar\alpha}^i\delta(x-y) - P_0^i\delta_{,\bar\alpha}(x-y) = 0,$$

which becomes after integration with respect to the y variable

$$\frac{1}{12}WQ^{\frac{1}{3}}Q_{\bar\alpha}^i + P_{0,\bar\alpha}^i = 0. \tag{62}$$

We assume fixed boundary conditions so that the variation on the boundary is zero and thus the boundary term is zero. Equation (62) represents 12 secondary constraints on the theory.

Before proceeding with the constraint algorithm a comment is in order. Computation of $P^i{}_{0,0} = [P_0^i, H_p]$, yields $P^i{}_{0,0} = -\frac{1}{12}WQ^{\frac{1}{3}}Q_0^i$. Hence, we have $\frac{1}{12}WQ^{\frac{1}{3}}Q_\alpha^i + P^i{}_{0,\alpha} = 0$ for $\alpha = 0, 1, 2, 3$. Since $Q^{\frac{1}{3}}Q_\alpha^i = h_\alpha^i$ and $P_0^i = -2C^i$ and $W = 4C^kC_k$ we see that Eq. (62) along with the dynamics for P_0^i imply that $C^i_{,j} = \frac{1}{6}\delta_j^iC^kC_k$ which is the Latin form of the field equations.

Proceeding with the algorithm we note that multiplication of Eq. (62) by $Q_i^0(x)$ yields the condition

$$Q_i^0P^i{}_{0,\bar\alpha} = 0. \tag{63}$$

The computation of further constraints is rather tedious. It will be useful to note that $\frac{\delta Q_\alpha^i}{\delta Q_j^\beta} = -Q_\beta^iQ_\alpha^j$. Because of Eq. (63) we must require that

$$\begin{aligned}0 &= \left[Q_i^0P^i{}_{0,\bar\alpha}, H_p\right]\\ &= P^i{}_{0,\bar\alpha}\left(\frac{1}{2}g_{ij}P_0^jQ^{\frac{1}{3}} - Q^{\bar\gamma}_{i,\bar\gamma}\right)\delta - Q_i^0(x)\delta_{,\bar\alpha}(x-y)\cdot\left(\frac{1}{2}g_{jk}P_0^jP_0^kQ^{\frac{1}{3}}Q_0^i\right)(y)\end{aligned}$$

After the integration with respect to y we have

$$\begin{aligned}0 &= \frac{1}{2}g_{ij}P^{i}{}_{0,\bar{\alpha}}P_0^j Q^{\frac{1}{3}} - P^{i}{}_{0,\bar{\alpha}}Q^{\bar{\gamma}}{}_{i,\bar{\gamma}} + \frac{1}{12}\left(g_{jk}P_0^j P_0^k Q^{\frac{1}{3}}Q_0^i\right)_{,\bar{\alpha}} Q_i^0 \\ &= \frac{2}{3}g_{ij}P^{i}{}_{0,\bar{\alpha}}P_0^j Q^{\frac{1}{3}} - P^{i}{}_{0,\bar{\alpha}}Q^{\bar{\gamma}}{}_{i,\bar{\gamma}} + \frac{1}{36}WQ^{\frac{1}{3}}Q_\beta^k Q^{\beta}_{k,\bar{\alpha}} + \frac{1}{12}WQ^{\frac{1}{3}}Q^{i}{}_{0,\bar{\alpha}}Q_i^0 \\ &= \frac{1}{36}WQ^{\frac{1}{3}}\left(-2g_{ij}Q_{\bar{\alpha}}^i P_0^j Q^{\frac{1}{3}} + 3Q_{\bar{\alpha}}^i Q^{\bar{\gamma}}{}_{i,\bar{\gamma}} + Q_\beta^i Q^{\beta}_{i,\bar{\alpha}} - 3Q_0^i Q^0_{i,\bar{\alpha}}\right).\end{aligned}$$

In the last line, the secondary constraint Eq. (62) has been used. Since $Q = 0$ is not acceptable, we find that

$$W = 0 \quad \text{or} \quad -2g_{ij}Q_{\bar{\alpha}}^i P_0^j Q^{\frac{1}{3}} + 3Q_{\bar{\alpha}}^i Q^{\bar{\gamma}}{}_{i,\bar{\gamma}} + Q_\beta^i Q^{\beta}_{i,\bar{\alpha}} - 3Q_0^i Q^0_{i,\bar{\alpha}} = 0. \quad \text{(64a,b)}$$

Type I Regions. We define Type I regions as simply connected regions of the space-like surface σ where $W \equiv 0$. In Type I regions the secondary constraint Eq. (62) implies that $P^{i}{}_{0,\bar{\alpha}} = 0$ also. We then require that

$$0 = [W, H_p] = -2g_{ij}P_0^j \frac{\delta H_p}{\delta Q_i^0} = -2g_{ij}P_0^j \cdot \frac{1}{12}WQ^{\frac{1}{3}}Q_0^i$$

and since W is assumed to be zero, this is automatically satisfied. Thus in the $W = 0$ case, the algorithm terminates. The secondary constraints in this case may be summarized by the four conditions:

$$P_0^i = K^i, \tag{65}$$

where K^i is constant and lightlike. These represent four first class, secondary constraints.

Type II Regions. We define Type II regions as simply connected regions of σ where W is not identically zero. In this case we assume that $-2g_{ij}Q_{\bar{\alpha}}^i P_0^j Q^{\frac{1}{3}} + 3Q_{\bar{\alpha}}^i Q^{\bar{\gamma}}{}_{i,\bar{\gamma}} + Q_\beta^i Q^{\beta}_{i,\bar{\alpha}} - 3Q_0^i Q^0_{i,\bar{\alpha}}$ is identically zero. Returning to the constraint given by Eq. (62) we require that

$$\begin{aligned}0 &= \left[\frac{1}{12}WQ^{\frac{1}{3}}Q_{\bar{\alpha}}^i + P^i_{0,\bar{\alpha},}H_p\right] \\ &= -\frac{1}{36}WQ^{\frac{1}{3}}\big(2Q_0^i g_{kl}Q_{\bar{\alpha}}^k P_0^l Q^{\frac{1}{3}} - 3Q_0^i Q_{\bar{\alpha}}^k Q^{\bar{\gamma}}_{k,\bar{\gamma}} + 3Q^i_{\bar{\gamma}}Q_{\bar{\alpha}}^k v_k^{\bar{\gamma}} \\ &\quad + Q_{\bar{\alpha}}^i Q_0^k Q^{\bar{\gamma}}_{k,\bar{\gamma}} - Q_{\bar{\alpha}}^i Q_{\bar{\gamma}}^k v_k^{\bar{\gamma}} - Q_\gamma^k Q^{\gamma}_{k,\bar{\alpha}}Q_0^i - 3Q^{i}{}_{0,\bar{\alpha}}\big).\end{aligned}$$

If we assume that W and Q are not identically zero, then we have 12 tertiary constraints:

$$\begin{aligned}0 &= 2Q_0^i g_{kl}Q_{\bar{\alpha}}^k P_0^l Q^{\frac{1}{3}} - 3Q_0^i Q_{\bar{\alpha}}^k Q^{\bar{\gamma}}_{k,\bar{\gamma}} + 3Q^i_{\bar{\gamma}}Q_{\bar{\alpha}}^k v_k^{\bar{\gamma}} + Q_{\bar{\alpha}}^i Q_0^k Q^{\bar{\gamma}}_{k,\bar{\gamma}} \\ &\quad - Q_{\bar{\alpha}}^i Q_{\bar{\gamma}}^k v_k^{\bar{\gamma}} - Q_\gamma^k Q^{\gamma}_{k,\bar{\alpha}}Q_0^i - 3Q^{i}{}_{0,\bar{\alpha}}.\end{aligned} \tag{66}$$

By multiplication by appropriate factors of Q_i^β we may split these 12 equations as follows. Multiplication of Eq. (66) by Q_i^0 implies that $2g_{ij}Q_{\bar\alpha}^i P_0^j Q^{\frac{1}{3}} - 3Q_{\bar\alpha}^k Q^{\bar\gamma}{}_{k,\bar\gamma} - Q_{\bar\beta}^k Q^{\beta}{}_{k,\bar\alpha} + 3Q_0^k Q^0{}_{k,\bar\alpha} = 0$. This is equivalent to the three constraints given in Eq. (64b). Next, multiplication of Eq. (66) by $Q_i^{\bar\alpha}$ yields the single constraint

$$Q_0^k Q^{\bar\gamma}{}_{k,\bar\gamma} = 0. \tag{67}$$

Finally, multiplication of Eq. (66) by $Q_i^{\bar\beta}$ results in the conditions $Q_{\bar\alpha}^k v_k^{\bar\beta} - \frac{1}{3}\delta_{\bar\alpha}^{\bar\beta} Q_{\bar\gamma}^k v_k^{\bar\gamma} + Q_0^i Q_{i,\bar\alpha}^{\bar\beta} = 0$. Using Eq. (67) these nine equations are seen to be traceless and hence these eight equations may be used to reduce the number of unknown Lagrange multipliers from 12 to 4. The result is

$$v_k^{\bar\beta} = \lambda Q_k^{\bar\beta} + \lambda^{\bar\beta} Q_k^0 - Q_0^j Q^{\bar\beta}{}_{j,\bar\gamma} Q_k^{\bar\gamma}, \tag{68}$$

where λ and $\lambda^{\bar\beta}$ represent four arbitrary Lagrange multiplier functions. It follows from Eq. (67) that we must require

$$\begin{aligned}
0 &= \left[Q_0^k Q^{\bar\gamma}{}_{k,\bar\gamma}, H_p\right] \\
&= Q^{\bar\gamma}{}_{k,\bar\gamma} \frac{\delta Q_0^k}{\delta Q_l^\alpha} \left(\delta_0^\alpha \left(\frac{1}{2} g_{il} P_0^i Q^{\frac{1}{3}} - Q^{\bar\beta}{}_{l,\bar\beta}\right) + v_l^{\bar\alpha}\right) + Q_0^k \frac{\delta Q^{\bar\gamma}{}_{k,\bar\gamma}}{\delta Q_l^\alpha} v_l^{\bar\alpha} \\
&= Q^{\bar\gamma}{}_{k,\bar\gamma} \left(-Q_\alpha^k Q_0^l\right) \left(\delta_0^\alpha \left(\frac{1}{2} g_{il} P_0^i Q^{\frac{1}{3}} - Q_{l,\bar\beta}^{\bar\beta}\right) + v_l^{\bar\alpha}\right) \\
&+ Q_0^k(x) \delta_{\bar\alpha}^{\bar\gamma} \delta_k^l \delta_{,\bar\gamma}(x-y) v_l^{\bar\alpha}(y) \\
&= -Q^{\bar\gamma}{}_{k,\bar\gamma} Q_{\bar\alpha}^k Q_0^l v_l^{\bar\alpha} - Q_0^l v_{l,\bar\alpha}^{\bar\alpha},
\end{aligned}$$

where the constraints have been used and an integration by parts has been performed on the second term. Now using Eq. (68) we find

$$\lambda_{,\bar\beta}^{\bar\beta} = -\left(Q_{\bar\beta}^k Q^{\bar\gamma}{}_{k,\bar\gamma} + Q_0^k Q_{k,\bar\beta}^0\right) \lambda^{\bar\beta} + Q_0^k Q_{k\bar\gamma}^{\bar\beta} Q_0^l Q_{l,\bar\beta}^{\bar\gamma}. \tag{69}$$

This differential equation represents one condition on the 3 multipliers $\lambda^{\bar\beta}$. Finally we proceed from the constraint given in Eq. (64b).

$$\begin{aligned}
0 &= \left[-2g_{ij} Q_{\bar\alpha}^i P_0^j Q^{\frac{1}{3}} + 3Q_{\bar\alpha}^i Q_{i,\bar\gamma}^{\bar\gamma} + Q_{\bar\beta}^k Q^{\beta}{}_{k,\bar\alpha} - 3Q_0^i Q_{i,\bar\alpha}^0, H_p\right] \\
&= \left(-2g_{kl} P_0^l Q_\beta^k Q_{\bar\alpha}^j Q^{\frac{1}{3}} + \frac{2}{3} g_{kl} P_0^l Q_{\bar\alpha}^k Q_\beta^j Q^{\frac{1}{3}} + 3Q^{\bar\gamma}{}_{k,\bar\gamma} Q_\beta^k Q_{\bar\alpha}^j - 3Q_{\bar\alpha}^j \delta_{,\bar\beta}\right. \\
&\quad \left. + Q^{\gamma}{}_{k,\bar\alpha} Q_\beta^k Q_\gamma^j - Q_\beta^j \delta_{,\bar\alpha} - 3Q^0{}_{i,\bar\alpha} Q_\beta^i Q_0^j + 3Q_0^j \delta_{,\bar\alpha} \delta_\beta^0\right)
\end{aligned}$$

$$\cdot\left(\delta_0^\beta\left(\frac{1}{2}g_{jm}P_0^m Q^{\frac{1}{3}} - Q^{\bar{\gamma}}{}_{j,\bar{\gamma}}\right) + v_j^{\bar{\beta}}\right) - \frac{1}{6}g_{kl}Q_{\bar{\alpha}}^k Q_0^l g_{mn}P_0^m P_0^n Q^{\frac{2}{3}}.$$

Substituting for $v_j^{\bar{\beta}}$ using Eq. (68) and using the constraints Eqs. (62), (64b), and (67) yield

$$\begin{aligned}
0 = {} & g_{kl}Q_0^k P_0^l Q^{\frac{1}{3}}\left(Q_{\bar{\alpha}}^i Q^{\bar{\beta}}{}_{i,\bar{\beta}} - \frac{2}{3}Q_\beta^i Q^\beta{}_{i,\bar{\alpha}} - \frac{1}{2}Q_0^i Q^0{}_{i,\bar{\alpha}}\right) \\
& - \frac{1}{12}g_{kl}P_0^k P_0^l g_{ij}Q_{\bar{\alpha}}^i Q_0^j Q^{\frac{2}{3}} \\
& + 5Q^i{}_{0,\bar{\alpha}}Q_{i,\bar{\beta}}^{\bar{\beta}} - 3Q_{0,\bar{\beta}}^i Q^{\bar{\beta}}{}_{i,\bar{\alpha}} + 2g_{kl}Q_{\bar{\beta}}^k P_0^l Q_0^i Q^{\bar{\beta}}{}_{i,\bar{\alpha}}Q^{\frac{1}{3}} \\
& - Q_0^i Q_{i,\bar{\gamma}}^{\bar{\beta}}\left(3Q_{\bar{\alpha}}^j Q^{\bar{\gamma}}{}_{j,\bar{\beta}} + 2Q_{\bar{\beta}}^j Q^{\bar{\gamma}}{}_{j,\bar{\alpha}}\right) - \frac{1}{2}g_{kl}P_0^k Q^l{}_{0,\bar{\alpha}}Q^{\frac{1}{3}} \\
& + \left(6Q_{\bar{\alpha}}^i Q^{\bar{\beta}}{}_{i,\bar{\beta}} + 2Q_{\bar{\beta}}^i Q^{\bar{\beta}}{}_{i,\bar{\alpha}}\right)\lambda + \left(3Q_{\bar{\alpha}}^i Q_{i,\bar{\beta}}^0 - Q_{\bar{\beta}}^i Q^0{}_{i,\bar{\alpha}}\right)\lambda^{\bar{\beta}} + 6\lambda_{,\bar{\alpha}}. \quad (70)
\end{aligned}$$

These three equations along with Eq. (69) may be used to solve for λ and the $\lambda^{\bar{\beta}}$ and since these are first-order differential equations in the lambdas, we expect that there will be four arbitrary constants in our solutions for the Lagrange multipliers. This completes the Dirac–Bergmann algorithm. For a summary see Table I.

For tetrads that satisfy the field equations we may check to determine whether the tetrad also agrees with the results of the Dirac–Bergmann algorithm. For tetrads with $C^i = 0$ or C^i constant and lightlike these results are clearly consistent and the region is Type I. When C^i is nonconstant and the field equations are satisfied it is not so obvious because all the tertiary constraints must be checked. For the example

Table I. Results of Constraint Algorithm

Case	$W = 0$	$W \neq 0$
Primary constraints	$P_{\bar{\alpha}}^i = 0$ (12 first class)	$P_{\bar{\alpha}}^i = 0$ (12 second class)
Secondary constraints	$P_0^i = K^i$, with constant K^i lightlike (4 first class)	$\frac{1}{12}WQ^{\frac{1}{3}}Q^{\bar{\alpha}} + P_{0,\bar{\alpha}}^i = 0$ (12 second class)
Tertiary constraints	None	$Q_0^k Q_{k,\bar{\beta}}^{\bar{\beta}} = 0$ and $2g_{ij}Q_{\bar{\alpha}}^i P_0^j Q^{\frac{1}{3}} - 3Q_{\bar{\alpha}}^k Q_{k,\bar{\beta}}^{\bar{\beta}} - Q_\beta^k Q_{k,\bar{\alpha}}^\beta + 3Q_0^k Q_{k,\bar{\alpha}}^0 = 0$ (4 second class)
Gauge fixing constraints	16 required	Gauge fixed by constraint algorithm
Degrees of freedom	0	4

given in Eq. (51), one finds that the tertiary constraints are indeed satisfied and the solutions for the Lagrange multipliers are $\lambda = -\frac{16x^0}{7\phi} + \frac{k^0 x^0}{\phi^6}$ and $\lambda^{\bar{\alpha}} = \kappa^{\bar{\alpha}}\phi^6$, where $\phi = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$, and κ^α are four arbitrary constants.

At present we do not have a physical interpretation of all the constraints. Recall that Gauss's law shows up as one of the constraints in the free electromagnetic field (Dirac, 1964). We expect that our secondary constraints will also have a similarly important interpretations.

In the case of Type I regions with $P_0^i = 0$ we see that the Hamiltonian is consistent with what is expected in a theory that describes gravitation. Multiplication of Eq. (62) by $Q_i^{\bar{\alpha}}$ implies that $\frac{1}{4}WQ^{\frac{1}{3}} + Q_i^{\bar{\alpha}} P^i{}_{0,\bar{\alpha}} = 0$. By comparison to Eq. (61) we see that H_p is weakly zero if $P_0^i = -2C^i$ is zero (Type I). Many investigators (e.g., Misner, 1957) expect that the correct Hamiltonian for gravity should be weakly zero.

Of the two types of regions, it would seem that the Type I regions would be more physically relevant. The 16 first class constraints would correspond to 16 gauge degrees of freedom. All the constraints are first class so that there is no need for the Dirac bracket. The Type II regions, however, have no gauge degrees of freedom and the Dirac bracket would be needed to define the symplectic structure on the 4-dimensional phase space.

4. CONCLUDING REMARKS

4.1. Possible Inclusion of the Strong Interaction

It may be possible to extend our theory to include the strong interaction, by replacing the real orthonormal tetrad h_μ^i with a complex orthonormal tetrad Z_μ^i which is restricted so that the space–time metric

$$g_{\mu\nu} = g_{ij}\overline{Z_\mu^i} Z_\nu^j \tag{71}$$

remains *real*. A bar indicates complex conjugation. That there exist complex tetrads which yield real metrics may be seen in the following way. It is known (see e.g., Barut, 1980) that there exist two complex groups which preserve the canonical Lorentz metric. One of these groups has complex transformation coefficients t_m^i which satisfy the relation $g_{mn} = g_{ij} t_m^i t_n^j$ where $g_{ij} = g_{mn} = \text{diag}(-1, 1, 1, 1)$. This group does not contain $SU(3)$ as a subgroup, and hence is of no interest here. The other group has complex transformation coefficients T_m^i which satisfy the relation

$$g_{mn} = g_{ij}\overline{T_m^i} T_n^j \tag{72}$$

where a bar indicates complex conjugation. This group contains $SU(3)$ as a proper subgroup. The components of the complex tetrad $Z_\mu^i = T_m^i h_\mu^m$ are complex valued

functions of the real space–time coordinates x^α. The complex conjugate of Z^i_μ is just $\overline{Z^i_\mu} = \overline{T^i_m} h^m_\mu$ because h^i_μ remains real. It is easily seen from Eq. (72) that Eq. (73) yields the same (real) metric as Eq. (1), i.e., $g_{\mu\nu} = g_{ij} h^i_\mu h^j_\nu$. Just as the real tetrad h^i_μ provides a richer structure than $g_{\mu\nu}$ (a structure which describes the gravitational and electroweak fields), the complex tetrad Z^i_μ provides an even richer structure (a structure which offers the possibility for describing the strong interaction, while still describing gravity with the real metric of general relativity).

4.1.1. Currents for Strong Isospin and Hypercharge

Working by analogy with Eq. (40), we define $\mathfrak{C}_i$ by

$$\mathfrak{C}_i == -Z^{-1}\left(Z Z^\nu_i\right)_{,\nu}, \tag{73}$$

where Z is the determinant of Z^i_μ, and note that $g^{ij}\overline{\mathfrak{C}}_i \mathfrak{C}_j$, is invariant not only under real conservative coordinate transformations on Greek indices, but also under complex conservative Lorentz transformations on Latin indices, i.e., transformations $Z^{\tilde{m}}_\mu = L^{\tilde{m}}_i Z^i_\mu$ which satisfy

$$L^j_{\tilde{m}} \left(L^{\tilde{m}}{}_{i,j} - L^{\tilde{m}}{}_{j,i}\right) = 0 \tag{74}$$

and $g_{ij} = g_{\tilde{m}\tilde{n}} \overline{L^{\tilde{m}}}_i L^{\tilde{n}}_j$ where $g_{ij} = g_{\tilde{m}\tilde{n}} = \mathrm{diag}(-1, 1, 1, 1)$. For an infinitesimal complex Latin Lorentz transformation, one easily finds that $L^i_{\tilde{m}} = \delta^i_m + g^{ij} \in_{jm}$ where $\in_{jm}$ is anti-Hermitian. The conservative condition, Eq. (74), is satisfied if and only if $\epsilon^i{}_{m,i} - \epsilon^i{}_{i,m} = 0$. From the ϵ^i_m one can read off the generators for the transformation coefficients $L^i_{\tilde{m}}$.

Field equations may be derived from a variational principle with Lagrangian $g^{ij}\overline{\mathfrak{C}}_i \mathfrak{C}_j$. The reality constraint on $g_{\mu\nu}$ is just

$$g_{ij}\left(\overline{Z^i_\mu} Z^j_\nu - Z^i_\mu \overline{Z^j_\nu}\right) = 0.$$

This constraint may be imposed by using Lagrange multipliers, and for the density h, we have $h = \sqrt{-g} = \sqrt{\overline{Z} Z}$. Thus, our variational principle is

$$\delta \int \left[g^{ij}\overline{\mathfrak{C}}_i \mathfrak{C}_j + \Lambda^{\mu\nu} g_{ij}\left(\overline{Z^i}{}_\mu Z^j_\nu - Z^i_\mu \overline{Z^j_\nu}\right)\right] \sqrt{\overline{Z} Z}\, d^4x = 0 \tag{75}$$

where Z^i_μ, $\overline{Z^i_\mu}$ and $\Lambda^{\mu\nu}$ are varied independently. (Independent variation of Z^i_μ and $\overline{Z^i_\mu}$ is equivalent to varying the real and imaginary components of Z^i_μ independently.)

After integration by parts, Noether's theorem gives (conserved) currents corresponding to strong isospin I_3 and hypercharge Y

$$\begin{aligned} I_3 &= \overline{C^1} Z_1^\alpha - C^1 \overline{Z_1^\alpha} - \overline{C^2} Z_2^\alpha + C^2 \overline{Z_2^\alpha} \\ Y &= \overline{C^1} Z_1^\alpha - C^1 \overline{Z_1^\alpha} + \overline{C^2} Z_2^\alpha - C^2 \overline{Z_2^\alpha} - 2\overline{C^3} Z_3^\alpha + 2C^3 \overline{Z_3^\alpha}. \end{aligned} \tag{76}$$

It is clear that our discussion of the strong interaction is more speculative than the discussions in previous sections. Much more work must be done before it may be possible to make a more definite claim.

The results presented in this paper indicate that this theory may lead to the fundamental theory that unifies all the known forces. The theory contains no adjustable parameters. The standard model, by contrast, requires that many parameter values and symmetries must be "put in by hand." The reason for this is that the standard model does not unify the electroweak and the strong interactions. And, of course gravity is not included in the standard model. In our theory, by contrast, gravity is present from the outset, and all forces are completely unified. Indeed, our theory is constructed by analogy with general relativity, while the $U(1) \times SU(2)$ electroweak theory and the $SU(3)$ strong theory (the building blocks of the standard model) are constructed by analogy with electromagnetism.

4.2. Quantization of the Theory

The theory thus far is at the classical level. Before quantization via canonical methods or path integrals, gauge constraints must be introduced to fix the gauge. Type I regions would require 16 gauge constraints, while none are required for Type II regions. Alternatively one may introduce 16 fermionic ghost variables and their conjugate momenta in Type I regions (Henneaux and Teitelboim, 1992; Sundermeyer, 1982; Weinberg, 1996). These extra degrees of freedom act as negative degrees of freedom which have the effect of fixing the gauge.

The quantized theory must be examined to determine whether it is finite, or, at least, renormalizable and free of anomalies. There are several reasons for believing that the quantized theory will be either finite or renormalizable. First, Rosenfeld (1930) noted certain advantages that tetrads present for the quantization of gravity. Second, our Lagrangian $C^\mu C_\mu$ involves only first derivatives of h_μ^i; whereas the Ricci scalar R, the Lagrangian for gravitation alone, involves first and second derivatives of $g_{\mu\nu}$. Third, the conservation group is much larger than the diffeomorphisms, and experience with gauge theory suggests that larger groups offer more promise of successful quantization. Fourth, we recall that the theory of weak interactions alone was not renormalizable, but the theory became renormalizable with the inclusion of electromagnetism. This provides hope that gravitation will become renormalizable with the inclusion of the electroweak and/or strong interaction.

4.3. Fundamental Geometrical Issues

It is possible that certain geometric principles could lead to a determination of coupling constants and masses. The larger symmetry of the conservation group suggests that the basic geometry is not a space of points, but a space of paths. Hence, we would investigate connections between this theory and string theory. It appears possible that the path-space could provide a geometrical foundation for string theory. The need for such a foundation has been emphasized especially by Witten (1988), and Schwarz (1988) has noted that this foundation could be provided by a "stringy space."

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